8. SOLONNIKOV V.A., Solvability of the spatial problem with a free boundary for a stationary system of Navier-Stokes equations, Zap. Nauchn. Seminarov LOMI, 84, 1979.
9. MOFFATT H.K., Viscous and resistive eddies near a sharp corner. J. Fluid Mech. 18, 1, 1964.

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HYDRODYNAMIC SINGULARITIES IN FLOWS WITH A FREE BOUNDARY*

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A method of determining the shape of the free surface of a planar stationary flow of a ponderable ideal fluid which flows around point hydrodynamic singularities is proposed. A Cauchy problem is formulated for finding the profile of such a flow. The self-induced motion of a point vortex under the free surface of an ideal ponderous fluid is considered.

1. *The equation of the profile of a capillary-gravitational wave of small amplitude on the surface of a stationary flow.* In the case of a stationary flow having point singularities, a method was proposed in /1/ for finding the shape of the free surface when it deviates slightly from the unperturbed position. The solution obtained by this method has the form of an improper integral with a variable limit. For example, in the case of the flow round a vortex of intensity Γ located at a depth h by a flow having a velocity $-V$ at positive infinity, the following expression /2/ can be obtained for the shape of the free surface:

$$S(x) = -\frac{\Gamma}{\pi V}\int\limits_{\infty}^{x} \frac{t\cos\nu(t-x) - h\sin\nu(t-x)}{t^2+h^2}\,dt \quad \left(\nu = \frac{g}{V^2}\right)$$

A still more complex integral representation for the function $S(x)$ can be obtained by this method when account is taken of capillary effects /3/.

Below, we obtain an ordinary differential equation which is satisfied by the function $S(x)$ and we formulate a Cauchy problem for determining it.

Let us consider a planar stationary flow with a velocity $-V$ at $x=\infty$. Let its unperturbed free surface coincide with the x-axis. Let us pick out the principal component of the flow by putting its complex potential equal to $W = \omega - Vz$, where $\omega = \varphi + i\psi$, $z = x + iy$. Linearized boundary conditions /3/

$$S(x) = (V/g)\,\varphi_x(x, 0) + [\alpha/(\rho g)]\,S''(x), \quad \psi(x, 0) = VS(x) \tag{1.1}$$

(α is the surface tension and ρ is the density of the fluid) can be written in the form of a single condition for the complex velocity $U(z) = w'(z)$ on the x-axis:

$$\operatorname{Im}(\beta U'' + iU' - \nu U) = 0 \quad (\beta = \alpha/(\rho V^2)) \tag{1.2}$$

Here and subsequently, a derivative of a function with respect to its argument is indicated by a prime.

If the flow passes around a unique singularity at the point $z_0 = -ih$, the complex velocity has the form $U(z) = C/(z+ih)^n + g(z)$, $n = 1, 2, \ldots$, where the function $g(z)$ is analytic over the whole of the domain of the flow. Following the method used in /1/, let us consider the function $f(z) = \beta U'' + iU' - \nu U$ which, as a consequence of condition (1.2), can be analytically extended according to the Schwartz principle into the upper half plane. Then,

$$f(z) = \beta F_+'' + iF_-' - \nu F_+, \quad F_\pm = C/(z+ih)^n \pm \bar{C}/(z-ih)^n$$

in the whole of the complex plane.

It can be directly verified that

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$$\beta^2 U'''' + (1 - 2\beta\nu) U'' + \nu^2 U = \beta f'' - if' - \nu f \tag{1.3}$$

It follows from the second boundary condition (1.1) that

$$R(x) \equiv S'(x) = V^{-1}\psi_x(x, 0) = V^{-1}\mathrm{Im}\, U|_{y=0}$$

which, when account is taken of (1.3), enables to write the equality

$$\beta^2 R'''' + (1 - 2\beta\nu) R'' + \nu^2 R = V^{-1}\,\mathrm{Im}\,(\beta f'' - if' - \nu f)|_{y=0}$$

By integrating both sides of this equation with respect to x and allowing for the fact that $\mathrm{Im}\, f = 0$ and $\mathrm{Im}\, f'' = 0$ when $y = 0$, we obtain the equation

$$\beta^2 S'''' + (1 - 2\beta\nu) S'' + \nu^2 S = -V^{-1} f(x) \tag{1.4}$$

in which the integration constant is omitted since it has no effect on the unique establishment of $S(x)$, which is only possible when additional conditions are specified. Let us formulate such conditions for the two important limiting cases of Eq.(1.4).

Suppose we are dealing with the flow of a fluid in which the action of capillary forces can be neglected. In this case, we put $\alpha = 0$, after which (1.4) becomes

$$S'' + \nu^2 S = -V^{-1} f_1(x), \quad f_1(x) = iF_-'(x) - \nu F_+(x) \tag{1.5}$$

We know /3/ that gravitational waves develop behind a singularity and not a long way upstream from it. The unique solution of Eq.(1.5) which satisfies the radiation condition

$$S(x) = -\frac{1}{V\nu}\int_{\infty}^{x} f_1(t)\sin\nu(x - t)\, dt \tag{1.6}$$

is the same as the solution of the problem obtained by the well-known method /1/.

In carrying out actual calculations, it is more convenient to deal not with the radiation conditions but with the Cauchy problem, which we shall now formulate. From relation (1.6) it is possible to find

$$S(0) = -\frac{2}{V}\,\mathrm{Re}\left[\frac{i^n C}{(n-1)!}\frac{d^{n-1}}{dh^{n-1}}\{e^{-\nu h}[\mathrm{Ei}(\nu h) + i\pi]\}\right] \tag{1.7}$$

$$S'(0) = \frac{2}{V}\,\mathrm{Re}\left\{i^{n+1}C\left[\frac{(-1)^{n+1}}{h^n} - \frac{\nu}{(n-1)!}\frac{d^{n-1}}{dh^{n-1}}(e^{-\nu h}[\mathrm{Ei}(\nu h) + i\pi])\right]\right\}$$

Eq.(1.5) with conditions (1.7) is the Cauchy problem for determining the profile of a purely gravitational wave on the surface of a stationary flow.

In particular, the problem

$$S'' + \nu^2 S = [\Gamma/(\pi V)]\,[(x^2 - h^2)/(x^2 + h^2) - \nu h]/(x^2 + h^2)$$
$$S(0) = -[\Gamma/(\pi V)]\exp(-\nu h)\,\mathrm{Ei}(\nu h), \quad S'(0) = (\Gamma\nu/V)\exp(-\nu h)$$

corresponds to the flow round a vortex of intensity Γ while the Q-problem

$$S'' + \nu^2 S = [Q/(\pi V)]\, x\,[2h/(x^2 + h^2) + \nu]/(x^2 + h^2)$$
$$S(0) = (Q/V)\exp(-\nu h), \quad S'(0) = [Q/(\pi V)]\,(\nu\exp(-\nu h)\,\mathrm{Ei}(\nu h) - 1/h)$$

corresponds to the flow round a source of copiousness.

Now let capillarity play a fundamental role. This means that one may put $\nu = 0$ in Eq.(1.4):

$$\beta^2 S'''' + S'' = -V^{-1} f_2(x), \quad f_2(x) = \beta F_+''(x) + iF_-'(x) \tag{1.8}$$

Unlike gravitational waves, capillary waves develop towards the flow /3/. We therefore require that the function $S(x)$ with all of its derivatives should tend to zero as $x \to -\infty$. If, in Eq.(1.8), one takes $P \equiv S''$ as the new function, then its unique solution, which satisfies the designated radiation condition, will be

$$P = -\frac{1}{\beta V}\int_{-\infty}^{x} f_2(t)\sin\frac{x - t}{\beta}\, dt$$

From this expression we find the initial conditions for determining P

$$P(0)=\frac{2}{\beta V}\operatorname{Re}\left[\frac{i^nC}{(n-1)!}\frac{d^n}{dh^n}\left\{\exp\left(-\frac{h}{\beta}\right)\left[\operatorname{Ei}\left(\frac{h}{\beta}\right)-i\pi\right]\right\}\right]$$

$$P'(0)=\frac{2}{\beta^2V}\operatorname{Re}\left\{\frac{i^{n+1}C}{(n-1)!}\left[\frac{d^n}{dh^n}\left\{\exp\left(-\frac{h}{\beta}\right)\left[\operatorname{Ei}\left(\frac{h}{\beta}\right)-i\pi\right]\right\}-\frac{(-1)^n n!}{h^{n+1}}\right]\right\}$$

It is difficult, in the general case to formulate a Cauchy problem similar to that which has been considered starting out just from the radiation conditions. However, it is possible to make use of an integral representation of the function $S(x)$ in order to find the values of S, S', S'' and S''' at any fixed point such as $x=0$, for example, and thereby obtain the Cauchy data for the solution of Eq.(1.4). So, in the case of the flow round a point vortex of intensity Γ, a corresponding integral representation /3/ enables one to find

$$S(0)=B\left[e^{-\sigma h}\operatorname{Ei}(\sigma h)\right]_-^+,\quad S'(0)=\pi B\left[\sigma e^{-\sigma h}\right]_-^+$$

$$S''(0)=B\left[\sigma/h-\sigma^2e^{-\sigma h}\operatorname{Ei}(\sigma h)\right]_-^+,\quad S'''(0)=-\pi B\left[\sigma^3e^{-\sigma h}\right]_-^+$$

$$(B=\Gamma/(\pi V\sqrt{1-4\beta\nu}),\quad [\Phi(\sigma)]_-^+=\Phi(\sigma_+)-\Phi(\sigma_-),\ \sigma_\pm=(1\pm\sqrt{1-4\beta\nu})/(2\beta)$$

2. Selfinduced motion of a vortex under a free surface. Let a single point vortex of intensity Γ be located at a certain point $z_0=x_0+iy_0$ of a bounded occupied by an ideal fluid. This means that the complex potential of the flow has the form $W=\Phi+i\Psi=[\Gamma/(2\pi i)]\ln(z-z_0)+p(z,t)$, where the function $p(z,t)$ is analytic with respect to z over the whole of the flow domain. The occurrence of the term $p(z,t)$ in the expression for the complex potential is associated with the presence of boundaries (in an unbounded medium $p\equiv 0$) in the fluid. Since the point vortex does not act in its own right and, furthermore, is "frozen" into the medium, its motion is realized in accordance with the equation $d\bar{z}_0/dt=p_z(z_0,t)$.

Let us now formulate the problem of describing, in the small-wave approximation, the motion of a point vortex under the free surface of an infinitely deep fluid located in a uniform gravitational field. Let us put $p(z,t)=[\Gamma/(2\pi i)]\ln(z-\bar{z}_0)+\omega(z,t)$, then $W=W_0+\omega$, where $W_0=\Phi_0+i\Psi_0=[\Gamma/(2\pi i)]\ln[(z-z_0)(z-\bar{z}_0)]$ and the function $\omega=\varphi+i\psi$ is analytic everywhere in the flow domain.

A vortex of small intensity located at a sufficient depth will generate surface waves of small amplitude. In this case, the boundary conditions on the x-axis have the form /3/

$$\Phi_t+gS=0,\quad \Phi_y=S_t \tag{2.1}$$

($S=S(x,t)$ is the deviation of the free surface from its unperturbed position $y=0$). The condition /3/

$$\Phi_{tt}+g\Phi_y\,|_{y=0}=0 \tag{2.2}$$

is a consequence of (2.1).

If the state of the free surface and the position of the vortex at the initial instant of time $t=0$ are specified

$$S(x,0)=S_0(x),\quad S_t(x,0)=S_1(x),\quad z_0(0)=-ih \tag{2.3}$$

then, when account is taken of the fact that the relationships $\Phi_0=\operatorname{Re}W_0=\operatorname{Re}\{[\Gamma/(\pi i)]\ln|x-z_0|\}=0$, $\partial\Phi_0/\partial t=0$ and

$$\partial\Phi_0/\partial y=-\operatorname{Im}W_0'=(\Gamma/\pi)(x-x_0)/[(x-x_0)^2+y_0^2]$$

hold on the x-axis, we get from (2.1)-(2.3) the problem for the velocity potential φ

$$\begin{gathered}\Delta\varphi=0\\ \varphi_{tt}+g\varphi_y\,|_{y=0}=-(g\Gamma/\pi)(x-x_0)/[(x-x_0)^2+y_0^2]\\ \varphi_y\,|_{y=0,\,t=0}=S_1(x)-(\Gamma/\pi)x/(x^2+h^2),\quad \varphi_t\,|_{y=0,\,t=0}=-gS_0(x)\end{gathered} \tag{2.4}$$

The form

$$\varphi=-\operatorname{Re}\left\{(\Gamma/\pi)\left(-i\ln[z-\overline{z_0(t)}]+\int_0^\infty\int_0^t\frac{d\overline{z_0(\xi)}}{d\xi}\exp(i\lambda[\overline{z_0(\xi)}-z])\times\right.\right.$$
$$\left.\left.\cos[\sqrt{g\lambda}(t-\xi)]\,d\xi\,d\lambda\right)\right\}+\Theta(x,y,t) \tag{2.5}$$

$$\Theta(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{\sigma_1(\lambda)}{|\lambda|}\cos(\sqrt{g|\lambda|}\,t)-\sqrt{\frac{g}{|\lambda|}}\,\sigma_0(\lambda)\sin(\sqrt{g|\lambda|}\,t)\right]\times$$

$$\exp(|\lambda| y + i\lambda x)\, d\lambda$$

($\sigma_0(\lambda)$ and $\sigma_1(\lambda)$ are the Fourier images of $S_0(x)$ and $S_1(x)$ respectively). May be ascribed to the solution of (2.4) obtained using a Fourier transformation.

The expression for the complex potential of the whole flow follows from (2.5)

$$W(z,t) = [\Gamma/(2\pi i)]\{\ln[z - z_0(t)] - \ln[z - \overline{z_0(t)}]\} - \tag{2.6}$$
$$\frac{\Gamma}{\pi}\int_0^\infty\int_0^t \frac{d\overline{z_0(\xi)}}{d\xi}\exp(i\lambda[\overline{z_0(\xi)} - z])\cos[\sqrt{g\lambda}(t-\xi)]\, d\xi\, d\lambda + \chi(z,t)$$

where $\chi(z, t)$ is a function which is analytic in the domain of the flow and the real part of this function is $\Theta(x, y, t)$.

The first term on the right-hand side of (2.6) is the complex potential of the flow which is created by the vortex being considered and by a "mirror" vortex with respect to the x-axis of intensity $-\Gamma$. The second term is associated with the motion of the free surface, which is caused by the vortex and the third term with the development of the initial perturbations of the free surface.

If, at the initial instant of time, the free surface was unperturbed, that is, it coincided with the x-axis and did not move, then

$$\frac{d\overline{z_0(t)}}{dt} = \frac{\Gamma}{\pi}\left\{\frac{1}{4y_0(t)} + i\int_0^\infty \lambda\int_0^t \frac{d\overline{z_0(\xi)}}{d\xi}\exp(i\lambda[\overline{z_0(\xi)} - z_0(t)]) \times \right. \tag{2.7}$$
$$\left.\cos[\sqrt{g\lambda}(t-\xi)]\, d\xi\, d\lambda\right\}$$

Since the small-wave approximation is being considered, the effect of the "mirror" vortex dominates and the action of the free surface is of a corrective nature. As a consequence of this, an approximate law for the motion of the vortex can be found by putting $z_0(\xi) = V\xi - ih$, $V = -\Gamma/(4\pi h)$ in the integrand of (2.7) which corresponds to the motion of a vortex when the free surface is "frozen", that is, in the case of a solid wall. The integration with respect to ξ in (2.7) is then carried out exactly, which yields

$$y_0(t) = -h + I_1, \quad x_0(t) = \frac{\Gamma}{4\pi}\int_0^t \frac{d\xi}{y_0(\xi)} - I_2$$

$$I_j = \frac{\Gamma^2}{8\pi^2 h}\int_0^\infty \lambda\exp(-2h\lambda)[F_j(\mu_+) + F_j(\mu_-)]\, d\lambda$$

$$(F_1(\mu) = (1 - \cos\mu t)/\mu^2, \quad F_2(\mu) = (\mu t - \sin\mu t)/\mu^2, \quad \mu_\pm = \lambda V \pm \sqrt{g\lambda})$$

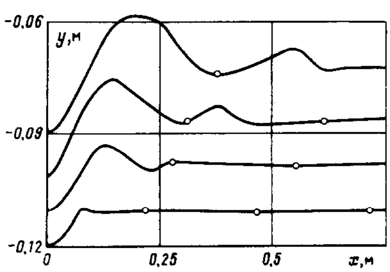

Fig.1

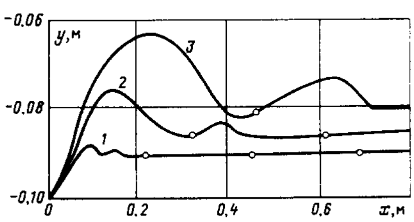

Fig.2

The trajectories of vortices of an intensity $\Gamma = -0.1\pi$ m^2/s located at the initial instant of time at depths of 9, 10, 11 and 12 cm, found according to these formula using a digital computer, are shown in Fig.1. The trajectories of vortices with intensities of -0.08π, -0.10π and -0.12π m^2/s located at a depth of 10 cm at the initial instant of time are shown in Fig.2 (curves *1*, *2* and *3* respectively). The points which have been picked out on the curves mark the positions of the vortices after each second. It can be seen that, after the oscillations, the greater the amplitude, the smaller the initial depth and the greater the intensity and the vortex reaches a state of monotonic and extremely slow leviation.

REFERENCES

1. KELDYSH M.V., Remarks on certain motions of a heavy fluid, in: M.V. Keldysh. Selected Papers. Mechanics (M.V. Keldysh. Izbrannye trudy, Mekhanika), Nauka, Moscow, 1985.

2. KOCHIN N.E., KIBEL I.A. and ROSE N.V., Theoretical Hydromechanics, 1, Gostekhizdat Moscow-Leningrad, 1948.
3. SRETENSKII L.N., Theory of the Wave Motions of a Fluid, Nauka, Moscow, 1977.

Translated by E.L.S.

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WAVE MOTIONS CAUSED BY A SOURCE IN A FLUID OF VARIABLE DEPTH*

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Solutions of the problem of the wave motion produced by a pulsed source moving in a fluid over an inclined bottom are obtained. An asymptotic analysis of the solution is carried out and the structures of the wave fields are investigated.

The motion of a source in a fluid of constant depth has been quite thoroughly studied by the successive application of integral transforms and the stationary phase method /1-3/. An asymptotic theory of wave motions has been developed /4/ for small variations of a base of arbitrary form. This is based on the use of the apparatus of pseudodifferential operators and the reduction of the problem to the solution of Hamiltonian systems. Only some special cases have been considered when there are significant changes in depth (the fluid is bounded by a planar inclined bottom): the problem has been formulated of the structure of the wave wake behind a moving source and a method of solving it has been pointed out in /5/, and a solution of the planar problem for a pulsating source has been constructed in /6/.

1. Let a source of intensity b, pulsating at a frequency ω and moving at a velocity c parallel to the shore line be placed in a fluid which occupies a wedge-shaped domain at the instant of time $t = 0$ (Fig.1).

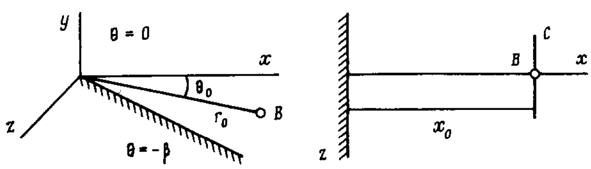

Fig.1

We shall write the equations, the boundary conditions and the initial conditions of the problem within the framework of linear dispersion theory /2, 7/

$$\begin{gathered}\Delta G = -b\,(4\pi r)^{-1}\,\delta\,(r - r_0, \theta - \theta_0, z)\, E\,(-\omega t) \\ G_{tt} + 2cG_{tz} + c^2G_{zz} + gr^{-1}G_\theta = 0, \quad \theta = 0 \\ G_\theta = 0, \quad \theta = -\beta; \quad G = G_t = 0, \quad \theta = 0, \quad t = 0 \\ G < \infty, \quad r \to 0; \quad G \to 0, \quad \sqrt{r^2 + z^2} \to \infty; \quad \eta = -g^{-1}G_t\,|_{\theta=0} \\ \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}, \quad \beta = \frac{\pi}{2n}, \quad n = 2m+1, m = 0,1,2,\ldots \\ E\,(\varkappa) = \exp\,(i\varkappa)\end{gathered} \tag{1.1}$$

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